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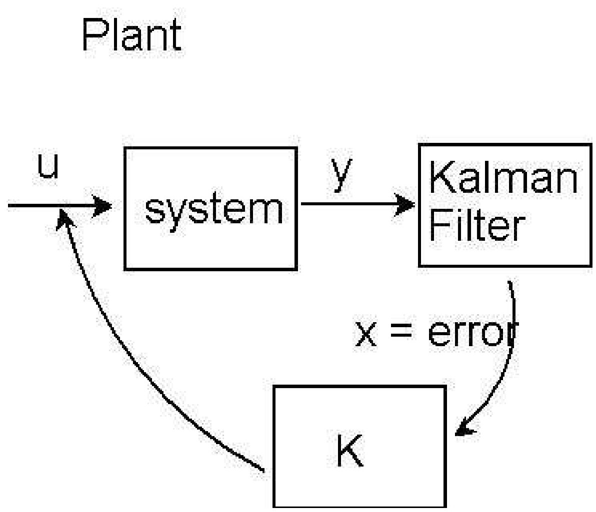
Basic Feedback controller Design

First let us consider this system:



With this system, our goal is to provide the smallest constant feedback K to u that would minimize the error x.

Assumption:

(C, A) is detectable, (A, B) is stabilitzble, $D^*D > 0$ and

$$\begin{pmatrix} A - j\omega I & B \\ C & D \end{pmatrix}$$

has full rank for all $\omega < \mathbb{R}$

P in the lyapunov's equation is positive definite.

Problem.

The system is described with this equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Since we are trying to minimize x and u, let's form another equation which contains them.

$$z = \begin{pmatrix} Q^{1/2}x \\ R^{1/2}u \end{pmatrix} = \begin{pmatrix} Q^{1/2} \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ R^{1/2} \end{pmatrix} u$$

this can also be written as

$$z = C_m x + D_m u$$

to minimize z the system we attemp to solve becomes

$$\dot{x} = Ax + Bu$$
$$z = C_m x + D_m u$$

From the figure above we state that

$$u = Kx$$

If we substitute u back into the the state equation 1 we get

$$\dot{x} = (A + BK)x \qquad \text{equation 3}$$
$$z = (C_m + D_m K)x$$

From this we can solve for x and z

$$x = e^{(A+BK)t} x_0 \qquad \text{equation 4}$$
$$z = (C_m + D_m K) e^{(A+BK)t} x_0$$

Now we are finally ready to minimize z by defining the size of z as the euclidian norm.

we want to minimize this value

$$\|z\|^2 = \int_0^\infty x_0^* e^{(A+BK)^* t} (C_m + D_m K)^* (C_m + D_m K) e^{(A+BK)t} x_0 \, dt$$

if we rewrite the equation, we get this form

$$\|z\|^2 = x_0^* \left[\int_0^\infty e^{(A+BK)^* t} (C_m + D_m K)^* (C_m + D_m K) e^{(A+BK)t} \, dt \right] x_0$$

we see that the integral is the solution to a variant of lyapunov equation where

$$\int_0^\infty e^{A^* t} M e^{A t} dt = \mathrm{P}$$

where P is the solution to the lyapunov equation:

$$A^* P + PA + M = 0$$

in our case

A would become $(A + BK)$

M would become $(\mathrm{C_m} + \mathrm{D_m}\,\mathrm{K})^* (\mathrm{C_m} + \mathrm{D_m}\,\mathrm{K})$

and the lyapunov equation would become:

$$(A + BK)^*\mathrm{P} + \mathrm{P}(A + BK) + (\mathrm{C_m} + \mathrm{D_m}\,\mathrm{K})^* (\mathrm{C_m} + \mathrm{D_m}\,\mathrm{K}) = 0$$

and z would become

$$\|z\|^2 = x_0^* P x_0$$

Remember that our goal is to minimize the euclidian magnitude of z so we naturally would want to find the optimum K inside P that would make P as small as possible. So in the next part our goal is to study the lyapunov equation carefully to find the K that would make P as small as possible.

So now that we have the lyapunov equation

$$(A + BK)^*\mathrm{P} + \mathrm{P}(A + BK) + (\mathrm{C_m} + \mathrm{D_m}\,\mathrm{K})^* (\mathrm{C_m} + \mathrm{D_m}\,\mathrm{K}) = 0$$

we can expand out all the terms and get.

$$A^* P + K^* B^* P + PA + PBK + C_m^* C_m + K^* D_m^* C_m + C_m^* D_m K + K^* D_m^* D_m K = 0$$

this expression can be rewritten as

$$K^* D_m^* D_m K + K^* \left[D_m^* C_m + B^* P\right] + \left[C_m^* D_m + PB\right] K + C_m^* C_m + A^* P + PA = 0$$

this is where complete the square comes in

$$(D_m K \ + \ (D_m^{-1^*} B^* P + C_m))^* (D_m K + (D_m^{-1^*} B^* P + C_m)) - PBD_m^{-1} D_m^{-1^*} B^* P$$
$$- C_m^* D_m^{-1^*} B^* P - PBD_m^{-1} C_m$$

If you find that P is not done with the proper complex conjugate, it is because P is a strictly positive matrix. So flipping the sign doesn't make a difference.

As it turns out, this equation looks like a bowl in a multidimensional space and the bottom of the bowl, or the minimum is at the point where

$$(\mathrm{D_m K} \ + (\mathrm{D}_m{}^{-1}\mathrm{BP}+\mathrm{C}_m)) = 0$$

Meaning that the optimum K is

$$K = -(D^* D)^{-1} (PB + C^* D)^*$$

Since $(\mathrm{D_m K} \ + (\mathrm{D}_m{}^{-1}\mathrm{BP}+\mathrm{C}_m)) = 0$, the lyapunov equation breaks down into Riccati equation:

$$A^* P + PA + C^* C - (PB + C^* D)(D^* D)^{-1} (PB + C^* D)^* = 0$$

You would first solve the Riccati's equation and then use the P from Riccati's equation to solve for K.